

BOUNDS FOR THE RATIO OF TWO GAMMA FUNCTIONS—FROM WENDEL’S LIMIT TO ELEZOVIĆ-GIORDANO-PEČARIĆ’S THEOREM

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ABSTRACT. In the survey paper, along one of main lines of bounding the ratio of two gamma functions, we look back and analyse some known results, including Wendel’s, Gurland’s, Kazarinoff’s, Gautschi’s, Watson’s, Chu’s, Lazarević-Lupaş’s, Kershaw’s and Elezović-Giordano-Pečarić’s inequalities, claim, monotonic and convex properties. On the other hand, we introduce some related advances on the topic of bounding the ratio of two gamma functions in recent years.

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1. INTRODUCTION

Recall [21, Chapter XIII] and [46, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \quad (1.1)$$

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for $x \in I$ and $n \geq 0$. The celebrated Bernstein-Widder Theorem [46, p. 160, Theorem 12a] states that a function f is completely monotonic on $[0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xs} d\mu(s), \quad (1.2)$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral (1.2) converges for all $x > 0$. This tells us that a completely monotonic function f on $[0, \infty)$ is a Laplace transform of the measure μ .

It is well-known that the classical Euler's gamma function may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \quad (1.3)$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. It is common knowledge that the special functions $\Gamma(x)$, $\psi(x)$ and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are fundamental and important and have much extensive applications in mathematical sciences.

The history of bounding the ratio of two gamma functions has been longer than at least sixty years since the paper [45] by J. G. Wendel was published in 1948.

The motivations of bounding the ratio of two gamma functions are various, including establishment of asymptotic relation, refinements of Wallis' formula, approximation to π , needs in statistics and other mathematical sciences.

In this survey paper, along one of main lines of bounding the ratio of two gamma functions, we would like to look back and analyse some known results, including Wendel's asymptotic relation, Gurland's approximation to π , Kazarinoff's refinement of Wallis' formula, Gautschi's double inequality, Watson's monotonicity, Chu's refinement of Wallis' formula, Lazarević-Lupaş's claim on monotonic and convex properties, Kershaw's first double inequality, Elezović-Giordano-Pečarić's theorem, alternative proofs of Elezović-Giordano-Pečarić's theorem and related consequences.

On the other hand, we would also like to describe some new advances in recent years on this topic, including the complete monotonicity of divided differences of the psi and polygamma functions, inequalities for sums and related results.

2. WENDEL'S DOUBLE INEQUALITY

Our starting point is a paper published in 1948 by J. G. Wendel, which is the earliest related one we could search out to the best of our ability.

In order to establish the classical asymptotic relation

$$\lim_{x \rightarrow \infty} \frac{\Gamma(x+s)}{x^s \Gamma(x)} = 1 \quad (2.1)$$

for real s and x , by using Hölder's inequality for integrals, J. G. Wendel [45] proved elegantly the double inequality

$$\left(\frac{x}{x+s} \right)^{1-s} \leq \frac{\Gamma(x+s)}{x^s \Gamma(x)} \leq 1 \quad (2.2)$$

for $0 < s < 1$ and $x > 0$.

Wendel's original proof. Let

$$\begin{aligned} 0 < s < 1, \quad p &= \frac{1}{s}, \quad q = \frac{p}{p-1} = \frac{1}{1-s}, \\ f(t) &= e^{-st} t^{sx}, \quad g(t) = e^{-(1-s)t} t^{(1-s)x+s-1}, \end{aligned}$$

and apply Hölder's inequality for integrals and the recurrent formula

$$\Gamma(x+1) = x\Gamma(x) \quad (2.3)$$

for $x > 0$ to obtain

$$\begin{aligned} \Gamma(x+s) &= \int_0^\infty e^{-t} t^{x+s-1} dt \\ &\leq \left(\int_0^\infty e^{-t} t^x dt \right)^s \left(\int_0^\infty e^{-t} t^{x-1} dt \right)^{1-s} \\ &= [\Gamma(x+1)]^s [\Gamma(x)]^{1-s} \\ &= x^s \Gamma(x). \end{aligned} \quad (2.4)$$

Replacing s by $1-s$ in (2.4) we get

$$\Gamma(x+1-s) \leq x^{1-s} \Gamma(x), \quad (2.5)$$

from which we obtain

$$\Gamma(x+1) \leq (x+s)^{1-s} \Gamma(x+s), \quad (2.6)$$

by substituting $x+s$ for x .

Combining (2.4) and (2.6) we get

$$\frac{x}{(x+s)^{1-s}} \Gamma(x) \leq \Gamma(x+s) \leq x^s \Gamma(x).$$

Therefore, the inequality (2.2) follows.

Letting x tend to infinity in (2.2) yields (2.1) for $0 < s < 1$. The extension to all real s is immediate on repeated application of (2.3). \square

Remark 1. The inequality (2.2) can be rewritten for $0 < s < 1$ and $x > 0$ as

$$x^{1-s} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq (x+s)^{1-s} \quad (2.7)$$

or

$$0 \leq \left[\frac{\Gamma(x+1)}{\Gamma(x+s)} \right]^{1/(1-s)} - x \leq s. \quad (2.8)$$

Remark 2. Using the recurrent formula (2.3) and the double inequality (2.7) repeatedly yields

$$x^{1-s} \frac{\prod_{i=1}^m (x+i)}{\prod_{i=0}^{n-1} (x+i+s)} \leq \frac{\Gamma(x+m+1)}{\Gamma(x+n+s)} \leq (x+s)^{1-s} \frac{\prod_{i=1}^m (x+i)}{\prod_{i=0}^{n-1} (x+i+s)} \quad (2.9)$$

for $x > 0$ and $0 < s < 1$, where m and n are positive integers. This implies that basing on the recurrent formula (2.3) and the double inequality (2.7) one can bound the ratio $\frac{\Gamma(x+a)}{\Gamma(x+b)}$ for any positive numbers x , a and b . Conversely, the double inequality (2.9) reveals that one can also deduce corresponding bounds of the ratio $\frac{\Gamma(x+1)}{\Gamma(x+s)}$ for $x > 0$ and $0 < s < 1$ from bounds of the ratio $\frac{\Gamma(x+a)}{\Gamma(x+b)}$ for positive numbers x , a and b .

Remark 3. In [1, p. 257, 6.1.46], the following limit was listed: For real numbers a and b ,

$$\lim_{x \rightarrow \infty} \left[x^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} \right] = 1. \quad (2.10)$$

The limits (2.1) and (2.10) are equivalent to each other since

$$x^{t-s} \frac{\Gamma(x+s)}{\Gamma(x+t)} = \frac{\Gamma(x+s)}{x^s \Gamma(x)} \cdot \frac{x^t \Gamma(x)}{\Gamma(x+t)}.$$

Hence, the limit (2.10) is called as Wendel's limit in the literature of this paper.

Remark 4. The double inequality (2.2) or (2.7) is more meaningful than the limit (2.1) or (2.10), since the former implies the latter, but not conversely.

Remark 5. Due to unknown reasons, Wendel's paper [45] and the double inequality (2.2) or (2.7) were seemingly neglected by nearly all mathematicians for more than fifty years until it was mentioned in [19], to the best of my knowledge.

3. GURLAND'S DOUBLE INEQUALITY

By making use of a basic theorem in mathematical statistics concerning unbiased estimators with minimum variance, J. Gurland [8] presented the following inequality

$$\left[\frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \right]^2 < \frac{n^2}{2n+1} \quad (3.1)$$

for $n \in \mathbb{N}$, and so taking respectively $n = 2k$ and $n = 2k+1$ for $k \in \mathbb{N}$ in (3.1) yields a closer approximation to π :

$$\frac{4k+3}{(2k+1)^2} \left[\frac{(2k)!!}{(2k-1)!!} \right]^2 < \pi < \frac{4}{4k+1} \left[\frac{(2k)!!}{(2k-1)!!} \right]^2, \quad k \in \mathbb{N}. \quad (3.2)$$

Remark 6. Taking respectively $n = 2k$ and $n = 2k-1$ for $k \in \mathbb{N}$ in (3.1) leads to

$$\sqrt{k + \frac{1}{4}} < \frac{\Gamma(k+1)}{\Gamma(k+1/2)} < \frac{2k}{\sqrt{4k-1}} = \frac{k}{\sqrt{k-1/4}}, \quad k \in \mathbb{N}. \quad (3.3)$$

This is better than the double inequality (2.7) for $x = k$ and $s = \frac{1}{2}$.

Remark 7. The double inequality (3.2) may be rearranged as

$$\sqrt{k + \frac{1}{4}} < \frac{\Gamma(k+1)}{\Gamma(k+1/2)} < \frac{2k+1}{\sqrt{4k+3}} = \frac{k+1/2}{\sqrt{k+1/2+1/4}}, \quad k \in \mathbb{N}. \quad (3.4)$$

It is easy to see that the upper bound in (3.4) is better than the corresponding one in (3.3). This phenomenon seemingly hints us that sharper bounds for the ratio $\frac{\Gamma(k+1)}{\Gamma(k+1/2)}$ can be obtained only if letting $m \in \mathbb{N}$ in $n = 2m-1$ is larger in (3.1). However, this is an illusion, since the lower bound of the following double inequality

$$\begin{aligned} \frac{k^2}{2k+1} \cdot \frac{\sqrt{4(k+m)-3}}{k+m-1} \prod_{i=0}^{m-1} \left[1 + \frac{1}{2(k+i)} \right] &< \frac{\Gamma(k+1)}{\Gamma(k+1/2)} \\ &< \left(\frac{2k}{2k-1} \right)^2 \frac{2(k+m)-3}{\sqrt{4(k+m)-5}} \prod_{i=0}^{m-1} \left[1 - \frac{1}{2(k+i)} \right], \end{aligned} \quad (3.5)$$

which is derived from taking respectively $n = 2(k+m-1)$ and $n = 2(k+m-1)-1$ for $k \in \mathbb{N}$ in (3.1), is decreasing and the upper bound of it is increasing with respect to m . Then how to explain the occurrence that the upper bound in (3.4) is stronger than the corresponding one in (3.3)?

Remark 8. The left-hand side inequality in (3.3) or (3.4) may be rearranged as

$$\frac{1}{4} < \left[\frac{\Gamma(k+1)}{\Gamma(k+1/2)} \right]^2 - k < \frac{1}{4} + \frac{1}{4(4k+3)}, \quad k \in \mathbb{N}. \quad (3.6)$$

From this, it is easier to see that the inequality (3.1) refines the double inequality (2.7) for $x = k$ and $s = \frac{1}{2}$.

Remark 9. It is noted that the inequality (3.1) was recovered in [6] and extended in [14] by different approaches respectively. See Section 4 and Section 7 below.

Remark 10. Just like the paper [45], Gurland's paper [8] was ignored except it was mentioned in [7, 41]. The famous monograph [20] recorded neither of the papers [8, 45].

4. KAZARINOFF'S DOUBLE INEQUALITY

Starting from one form of the celebrated formula of John Wallis:

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi n}}, \quad n \in \mathbb{N}, \quad (4.1)$$

which had been quoted for more than a century before 1950s by writers of textbooks, D. K. Kazarinoff proved in [14] that the sequence $\theta(n)$ defined by

$$\frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi[n+\theta(n)]}} \quad (4.2)$$

satisfies $\frac{1}{4} < \theta(n) < \frac{1}{2}$ for $n \in \mathbb{N}$. This implies

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}, \quad n \in \mathbb{N}. \quad (4.3)$$

Remark 11. It was said in [14] that it is unquestionable that inequalities similar to (4.3) can be improved indefinitely but at a sacrifice of simplicity, which is why the inequality (4.1) had survived so long.

Remark 12. Kazarinoff's proof of (4.3) is based upon the property

$$[\ln \phi(t)]'' - \{[\ln \phi(t)]'\}^2 > 0 \quad (4.4)$$

of the function

$$\phi(t) = \int_0^{\pi/2} \sin^t x \, dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma((t+1)/2)}{\Gamma((t+2)/2)} \quad (4.5)$$

for $-1 < t < \infty$. The inequality (4.4) was proved by making use of the well-known Legendre's formula

$$\psi(x) = -\gamma + \int_0^1 \frac{t^{x-1} - 1}{t-1} dt \quad (4.6)$$

for $x > 0$ and estimating the integrals

$$\int_0^1 \frac{x^t}{1+x} dx \quad \text{and} \quad \int_0^1 \frac{x^t \ln x}{1+x} dx. \quad (4.7)$$

Since (4.4) is equivalent to the statement that the reciprocal of $\phi(t)$ has an everywhere negative second derivative, therefore, for any positive t , $\phi(t)$ is less than the harmonic mean of $\phi(t-1)$ and $\phi(t+1)$, which implies

$$\frac{\Gamma((t+1)/2)}{\Gamma((t+2)/2)} < \frac{2}{\sqrt{2t+1}}, \quad t > -\frac{1}{2}. \quad (4.8)$$

As a subcase of this result, the right-hand side inequality in (4.3) is established.

Remark 13. Using the recurrent formula (2.3) in (4.8) gives

$$\left[\frac{\Gamma((t+1)/2)}{\Gamma(t/2)} \right]^2 < \frac{t^2}{2t+1} \quad (4.9)$$

for $t > 0$, which extends the inequality (3.1). This shows that Kazarinoff's paper [14] contains much general conclusions and that all results in [8] stated in Section 3 are consequences of the inequality (4.9), as showed below.

Replacing t by $2t$ in (4.8) or (4.9) and rearranging yield

$$\frac{\Gamma(t+1)}{\Gamma(t+1/2)} > \sqrt{t + \frac{1}{4}} \iff \left[\frac{\Gamma(t+1)}{\Gamma(t+1/2)} \right]^2 - t > \frac{1}{4} \quad (4.10)$$

for $t > 0$, which extends the left-hand side inequality in (3.3) and (3.4). Replacing t by $2t - 1$ in (4.8) or (4.9) produces

$$\frac{\Gamma(t+1)}{\Gamma(t+1/2)} < \frac{2t}{\sqrt{4t-1}} \quad (4.11)$$

for $t > \frac{1}{2}$, which extends the right-hand side inequality in (3.3). Replacing t by $2t + 1$ in (4.8) or (4.9) and rearranging gives

$$\frac{\Gamma(t+1)}{\Gamma(t+1/2)} < \frac{2t+1}{\sqrt{4t+3}} \quad (4.12)$$

for $t > -\frac{1}{2}$, which extends the right-hand side inequality in (3.4).

Remark 14. By the well-known Wallis cosine formula [44], the sequence $\theta(n)$ defined by (4.2) may be rearranged as

$$\theta(n) = \frac{1}{\pi} \left[\frac{(2n)!!}{(2n-1)!!} \right]^2 - n = \left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)} \right]^2 - n \quad (4.13)$$

for $n \in \mathbb{N}$. Then the inequality (4.3) is equivalent to

$$\frac{1}{4} < \left[\frac{\Gamma(n+1)}{\Gamma(n+1/2)} \right]^2 - n < \frac{1}{2}, \quad n \in \mathbb{N}. \quad (4.14)$$

Remark 15. The inequality (4.4) may be rewritten as

$$\psi' \left(\frac{t+1}{2} \right) - \psi' \left(\frac{t+2}{2} \right) > \left[\psi \left(\frac{t+1}{2} \right) - \psi \left(\frac{t+2}{2} \right) \right]^2 \quad (4.15)$$

for $t > -1$. Letting $u = \frac{t+1}{2}$ in the above inequality yields

$$\psi'(u) - \psi' \left(u + \frac{1}{2} \right) > \left[\psi(u) - \psi \left(u + \frac{1}{2} \right) \right]^2 \quad (4.16)$$

for $u > 0$. This inequality has been generalized in [28] to the complete monotonicity of a function involving divided differences of the digamma and trigamma functions as follows.

Theorem 1. For real numbers s, t , $\alpha = \min\{s, t\}$ and λ , let

$$\Delta_{s,t;\lambda}(x) = \begin{cases} \left[\frac{\psi(x+t) - \psi(x+s)}{t-s} \right]^2 + \lambda \frac{\psi'(x+t) - \psi'(x+s)}{t-s}, & s \neq t \\ [\psi'(x+s)]^2 + \lambda \psi''(x+s), & s = t \end{cases} \quad (4.17)$$

on $(-\alpha, \infty)$. Then the function $\Delta_{s,t;\lambda}(x)$ has the following complete monotonicity:

- (1) For $0 < |t-s| < 1$,
 - (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \leq 1$,
 - (b) so is the function $-\Delta_{s,t;\lambda}(x)$ if and only if $\lambda \geq \frac{1}{|t-s|}$;
- (2) For $|t-s| > 1$,
 - (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic on $(-\alpha, \infty)$ if and only if $\lambda \leq \frac{1}{|t-s|}$,
 - (b) so is the function $-\Delta_{s,t;\lambda}(x)$ if and only if $\lambda \geq 1$;
- (3) For $s = t$, the function $\Delta_{s,s;\lambda}(x)$ is completely monotonic on $(-s, \infty)$ if and only if $\lambda \leq 1$;
- (4) For $|t-s| = 1$,
 - (a) the function $\Delta_{s,t;\lambda}(x)$ is completely monotonic if and only if $\lambda < 1$,
 - (b) so is the function $-\Delta_{s,t;\lambda}(x)$ if and only if $\lambda > 1$,
 - (c) and $\Delta_{s,t;1}(x) \equiv 0$.

Taking in Theorem 1 $\lambda = s - t > 0$ produces that the function $\frac{\Gamma(x+s)}{\Gamma(x+t)}$ on $(-t, \infty)$ is increasingly convex if $s - t > 1$ and increasingly concave if $0 < s - t < 1$.

5. WATSON'S MONOTONICITY

In 1959, motivated by the result in [14] mentioned in Section 4, G. N. Watson [42] observed that

$$\begin{aligned} \frac{1}{x} \cdot \frac{[\Gamma(x+1)]^2}{[\Gamma(x+1/2)]^2} &= {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; x; 1\right) \\ &= 1 + \frac{1}{4x} + \frac{1}{32x(x+1)} + \sum_{r=3}^{\infty} \frac{[(-1/2) \cdot (1/2) \cdot (3/2) \cdot (r-3/2)]^2}{r!x(x+1) \cdots (x+r-1)} \end{aligned} \quad (5.1)$$

for $x > -\frac{1}{2}$, which implies the much general function

$$\theta(x) = \left[\frac{\Gamma(x+1)}{\Gamma(x+1/2)} \right]^2 - x \quad (5.2)$$

for $x > -\frac{1}{2}$, whose special case is the sequence $\theta(n)$ for $n \in \mathbb{N}$ defined in (4.2) or (4.13), is decreasing and

$$\lim_{x \rightarrow \infty} \theta(x) = \frac{1}{4} \quad \text{and} \quad \lim_{x \rightarrow (-1/2)^+} \theta(x) = \frac{1}{2}. \quad (5.3)$$

This apparently implies the sharp inequalities

$$\frac{1}{4} < \theta(x) < \frac{1}{2} \quad (5.4)$$

for $x > -\frac{1}{2}$,

$$\sqrt{x + \frac{1}{4}} < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \leq \sqrt{x + \frac{1}{4} + \left[\frac{\Gamma(3/4)}{\Gamma(1/4)} \right]^2} = \sqrt{x + 0.36423 \cdots} \quad (5.5)$$

for $x \geq -\frac{1}{4}$, and, by Wallis cosine formula [44],

$$\frac{1}{\sqrt{\pi(n+4/\pi-1)}} \leq \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}, \quad n \in \mathbb{N}. \quad (5.6)$$

In [42], an alternative proof of the double inequality (5.4) was also provided as follows: Let

$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^{\pi/2} \cos^{2x} t \, dt = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp(-xt^2) \frac{t \exp(-t^2/2)}{\sqrt{1 - \exp(-t^2)}} \, dt \quad (5.7)$$

for $x > \frac{1}{2}$. By using the fairly obvious inequalities

$$\sqrt{1 - \exp(-t^2)} \leq t \quad (5.8)$$

and

$$\frac{t \exp(-t^2/4)}{\sqrt{1 - \exp(-t^2)}} = \frac{t}{\sqrt{2 \sinh(t^2/2)}} \leq 1, \quad (5.9)$$

we have, for $x > -\frac{1}{4}$,

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} \exp(-(x+1/2)t^2) \, dt < f(x) < \frac{1}{\sqrt{\pi}} \int_0^{\infty} \exp(-(x+1/4)t^2) \, dt,$$

that is to say

$$\frac{1}{\sqrt{x+1/2}} < f(x) < \frac{1}{\sqrt{x+1/4}}. \quad (5.10)$$

Remark 16. It is easy to see that the inequality (5.5) extends and improves (2.7) if $s = \frac{1}{2}$, say nothing of (6.11) and (6.12) if $s = \frac{1}{2}$.

Remark 17. The left-hand side inequality in (5.6) is better than the corresponding one in (4.3) but worse than the corresponding one in (3.2) for $n \geq 2$.

Remark 18. The formula (5.1) implies the complete monotonicity of the function $\theta(x)$ defined by (5.2) on $(-\frac{1}{2}, \infty)$.

6. GAUTSCHI'S DOUBLE INEQUALITIES

The main aim of the paper [10] was to establish the double inequality

$$\frac{(x^p + 2)^{1/p} - x}{2} < e^{x^p} \int_x^\infty e^{-t^p} dt \leq c_p \left[\left(x^p + \frac{1}{c_p} \right)^{1/p} - x \right] \quad (6.1)$$

for $x \geq 0$ and $p > 1$, where

$$c_p = \left[\Gamma \left(1 + \frac{1}{p} \right) \right]^{p/(p-1)} \quad (6.2)$$

or $c_p = 1$.

By an easy transformation, the inequality (6.1) was written in terms of the complementary gamma function

$$\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt \quad (6.3)$$

as

$$\frac{p[(x+2)^{1/p} - x^{1/p}]}{2} < e^x \Gamma \left(\frac{1}{p}, x \right) \leq p c_p \left[\left(x + \frac{1}{c_p} \right)^{1/p} - x^{1/p} \right] \quad (6.4)$$

for $x \geq 0$ and $p > 1$. In particular, if letting $p \rightarrow \infty$, the double inequality

$$\frac{1}{2} \ln \left(1 + \frac{2}{x} \right) \leq e^x E_1(x) \leq \ln \left(1 + \frac{1}{x} \right) \quad (6.5)$$

for the exponential integral $E_1(x) = \Gamma(0, x)$ for $x > 0$ was derived from (6.4), in which the bounds exhibit the logarithmic singularity of $E_1(x)$ at $x = 0$.

As a direct consequence of the inequality (6.4) for $p = \frac{1}{s}$, $x = 0$ and $c_p = 1$, the following simple inequality for the gamma function was deduced:

$$2^{s-1} \leq \Gamma(1+s) \leq 1, \quad 0 \leq s \leq 1. \quad (6.6)$$

The second main result of the paper [10] was a sharper and more general inequality

$$e^{(s-1)\psi(n+1)} \leq \frac{\Gamma(n+s)}{\Gamma(n+1)} \leq n^{s-1} \quad (6.7)$$

for $0 \leq s \leq 1$ and $n \in \mathbb{N}$ than (6.6) by proving that the function

$$f(s) = \frac{1}{1-s} \ln \frac{\Gamma(n+s)}{\Gamma(n+1)} \quad (6.8)$$

is monotonically decreasing for $0 \leq s < 1$. Since $\psi(n) < \ln n$, it was derived from the inequality (6.7) that

$$\left(\frac{1}{n+1} \right)^{1-s} \leq \frac{\Gamma(n+s)}{\Gamma(n+1)} \leq \left(\frac{1}{n} \right)^{1-s}, \quad 0 \leq s \leq 1, \quad (6.9)$$

which was also rewritten as

$$\frac{n!(n+1)^{s-1}}{(s+1)(s+2)\cdots(s+n-1)} \leq \Gamma(1+s) \leq \frac{(n-1)!n^s}{(s+1)(s+2)\cdots(s+n-1)} \quad (6.10)$$

and so a simple proof of Euler's product formula in the segment $0 \leq s \leq 1$ was showed by letting $n \rightarrow \infty$ in (6.10).

Remark 19. The double inequalities (6.7) and (6.9) can be further rearranged as

$$n^{1-s} \leq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leq \exp((1-s)\psi(n+1)) \quad (6.11)$$

and

$$n^{1-s} \leq \frac{\Gamma(n+1)}{\Gamma(n+s)} \leq (n+1)^{1-s} \quad (6.12)$$

for $n \in \mathbb{N}$ and $0 \leq s \leq 1$.

Remark 20. The upper bounds in (2.7) and (6.11) have the following relationship

$$(n+s)^{1-s} \leq \exp((1-s)\psi(n+1)) \quad (6.13)$$

for $0 \leq s \leq \frac{1}{2}$ and $n \in \mathbb{N}$, and the inequality (6.13) reverses for $s > e^{1-\gamma} - 1 = 0.52620 \dots$, since the function

$$Q(x) = e^{\psi(x+1)} - x \quad (6.14)$$

was proved in [35, Theorem 2] to be strictly decreasing on $(-1, \infty)$, with

$$\lim_{x \rightarrow \infty} Q(x) = \frac{1}{2}. \quad (6.15)$$

This means that Wendel's double inequality (2.7) and Gautschi's first double inequality (6.11) are not included each other but they all contain Gautschi's second double inequality (6.12).

Remark 21. The right-hand side inequality in (6.11) may be rearranged as

$$\left[\frac{\Gamma(n+1)}{\Gamma(n+s)} \right]^{1/(1-s)} \leq \exp(\psi(n+1)), \quad n \in \mathbb{N}. \quad (6.16)$$

This suggests us the following double inequality

$$\exp(\psi(\alpha(x))) < \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} \leq \exp(\psi(\beta(x))) \quad (6.17)$$

for real numbers s, t and $x \in (-\min\{s, t\}, \infty)$, where $\alpha(x) \sim x$ and $\beta(x) \sim x$ as $x \rightarrow \infty$. For detailed information on the type of inequalities like (6.17), please refer to [26] and related references therein.

Remark 22. The inequality (6.12) can be rewritten as

$$0 \leq \left[\frac{\Gamma(n+1)}{\Gamma(n+s)} \right]^{1/(1-s)} - n \leq 1 \quad (6.18)$$

for $n \in \mathbb{N}$ and $0 \leq s \leq 1$.

Remark 23. In the texts of the reviews on the paper [10] by the Mathematical Reviews and the Zentralblatt MATH, there is no a word to comment on inequalities in (6.11) and (6.12). However, these two double inequalities later became a major source of a series of study on bounding the ratio of two gamma functions.

7. CHU'S DOUBLE INEQUALITY

In 1962, by discussing that

$$b_{n+1}(c) \geq b_n(c) \quad (7.1)$$

if and only if $(1-4c)n+1-3c \geq 0$, where

$$b_n(c) = \frac{(2n-1)!!}{(2n)!!} \sqrt{n+c}, \quad (7.2)$$

it was demonstrated in [6, Theorem 1] that

$$\frac{1}{\sqrt{\pi[n + (n+1)/(4n+3)]}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}, \quad n \in \mathbb{N}. \quad (7.3)$$

As an application of (7.3), by using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and (2.3), the following double inequality

$$\sqrt{\frac{2n-3}{4}} < \frac{\Gamma(n/2)}{\Gamma(n/2-1/2)} \leq \sqrt{\frac{(n-1)^2}{2n-1}} \quad (7.4)$$

for positive integers $n \geq 2$ was given in [6, Theorem 2].

Remark 24. After letting $n = 2k + 1$ the inequality (7.4) becomes

$$\sqrt{k - \frac{1}{4}} < \frac{\Gamma(k+1/2)}{\Gamma(k)} < \frac{k}{\sqrt{k+1/4}}, \quad (7.5)$$

which is same as (3.3). Taking $n = 2k + 2$ in leads to inequalities (3.4) and (3.6).

Notice that the reasoning directions in the two papers [6, 8] are opposite:

$$\frac{(2n-1)!!}{(2n)!!} \xrightarrow[\text{[8]}]{\text{[6]}} \frac{\Gamma(n/2)}{\Gamma(n/2-1/2)}. \quad (7.6)$$

To some extent, the results obtained by Gurland in [8] and by Chu in [6] are equivalent to each other and they are all special cases of those obtained by Kazarinoff in [14].

Remark 25. By Wallis cosine formula [44], the sequence (7.2) may be rewritten as

$$b_n(c) = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(n+1/2)}{\Gamma(n+1)} \sqrt{n+c} \triangleq \frac{1}{\sqrt{\pi}} B_c(n) \quad (7.7)$$

for $n \in \mathbb{N}$. Therefore, Chu discussed equivalently the necessary and sufficient conditions such that the sequence $B_c(n)$ for $n \in \mathbb{N}$ is monotonic.

Recently, necessary and sufficient conditions for the general function

$$H_{a,b,c}(x) = (x+c)^{b-a} \frac{\Gamma(x+a)}{\Gamma(x+b)} \quad (7.8)$$

on $(-\rho, \infty)$, where a, b and c are real numbers and $\rho = \min\{a, b, c\}$, to be logarithmically completely monotonic are presented in [37, 38]. A positive function f is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on I and its logarithm $\ln f$ satisfies $(-1)^k [\ln f(x)]^{(k)} \geq 0$ for $k \in \mathbb{N}$ on I , see [3, 4, 33].

8. LAZAREVIĆ-LUPAŞ'S CLAIM

In 1974, among other things, the function

$$\theta_\alpha(x) = \left[\frac{\Gamma(x+1)}{\Gamma(x+\alpha)} \right]^{1/(1-\alpha)} - x \quad (8.1)$$

on $(0, \infty)$ for $\alpha \in (0, 1)$ was claimed in [17, Theorem 2] to be decreasing and convex, and so

$$\frac{\alpha}{2} < \left[\frac{\Gamma(x+1)}{\Gamma(x+\alpha)} \right]^{1/(1-\alpha)} - x \leq [\Gamma(\alpha)]^{1/(1-\alpha)}. \quad (8.2)$$

Remark 26. The proof of [17, Theorem 2] is wrong, see [2, Remark 3.3] and [9, p. 240]. However, the statements in [17, Theorem 2] are correct and this is the first time to try to investigate the monotonic and convex properties of the much general function $\theta_\alpha(x)$.

9. KERSHAW'S FIRST DOUBLE INEQUALITY

In 1983, motivated by the inequality (6.12) obtained in [10], among other things, Kershaw presented in [15] the following double inequality

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x - \frac{1}{2} + \left(s + \frac{1}{4}\right)^{1/2}\right]^{1-s} \quad (9.1)$$

for $0 < s < 1$ and $x > 0$. In the literature, it is called as Kershaw's first double inequality for the ratio of two gamma functions.

Kershaw's proof for (9.1). Define the function g_β by

$$g_\beta(x) = \frac{\Gamma(x+1)}{\Gamma(x+s)}(x+\beta)^{s-1} \quad (9.2)$$

for $x > 0$ and $0 < s < 1$, where the parameter β is to be determined.

It is not difficult to show, with the aid of Wendel's limit (2.10), that

$$\lim_{x \rightarrow \infty} g_\beta(x) = 1. \quad (9.3)$$

To prove the double inequality (9.1) define

$$G(x) = \frac{g_\beta(x)}{g_\beta(x+1)} = \frac{x+s}{x+1} \left(\frac{x+\beta+1}{x+\beta} \right)^{1-s}, \quad (9.4)$$

from which it follows that

$$\frac{G'(x)}{G(x)} = \frac{(1-s)[(\beta^2 + \beta - s) + (2\beta - s)x]}{(x+1)(x+s)(x+\beta)(x+\beta+1)}.$$

This will leads to

- (1) if $\beta = \frac{s}{2}$, then $G'(x) < 0$ for $x > 0$;
- (2) if $\beta = -\frac{1}{2} + \left(s + \frac{1}{4}\right)^{1/2}$, then $G'(x) > 0$ for $x > 0$.

Consequently if $\beta = \frac{s}{2}$ then G strictly decreases, and since $G(x) \rightarrow 1$ as $x \rightarrow \infty$ it follows that $G(x) > 1$ for $x > 0$. But, from (9.3), this implies that $g_\beta(x) > g_\beta(x+1)$ for $x > 0$, and so $g_\beta(x) > g_\beta(x+n)$. Take the limit as $n \rightarrow \infty$ to give the result that $g_\beta(x) > 1$, which can be rewritten as the left-hand side inequality in (9.1). The corresponding upper bound can be verified by a similar argument when $\beta = -\frac{1}{2} + \left(s + \frac{1}{4}\right)^{1/2}$, the only difference being that in this case g_β strictly increases to unity. \square

Remark 27. The spirit of Kershaw's proof is similar to Chu's in [6, Theorem 1], as showed by (7.1). This idea or method was also utilized independently in [11, 12, 13, 16, 18, 22] to construct for various purposes a number of inequalities of the type

$$(x+\alpha)^{s-1} < \frac{\Gamma(x+s)}{\Gamma(x+1)} < (x+\beta)^{s-1} \quad (9.5)$$

for $s > 0$ and real number $x \geq 0$.

Remark 28. It is easy to see that the inequality (9.1) refines and extends the inequality (2.7), say nothing of (6.12).

Remark 29. The inequality (9.1) may be rearranged as

$$\frac{s}{2} < \left[\frac{\Gamma(x+1)}{\Gamma(x+s)} \right]^{1/(1-s)} - x < \left(s + \frac{1}{4} \right)^{1/2} - \frac{1}{2} \quad (9.6)$$

for $x > 0$ and $0 < s < 1$.

10. ELEZOVIĆ-GIORDANO-PEČARIĆ'S THEOREM

The inequalities (2.8), (3.6), (6.18) and (9.6), the sequence (4.13) and the function (5.2) and (8.1) strongly suggest us to consider the monotonic and convex properties of the general function

$$z_{s,t}(x) = \begin{cases} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} - x, & s \neq t \\ e^{\psi(x+s)} - x, & s = t \end{cases} \quad (10.1)$$

for $x \in (-\alpha, \infty)$, where s and t are two real numbers and $\alpha = \min\{s, t\}$.

In 2000, N. Elezović, C. Giordano and J. Pečarić gave in [9, Theorem 1] a perfect solution to the monotonic and convex properties of the function $z_{s,t}(x)$ as follows.

Theorem 2. *The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s| < 1$ or concave and increasing for $|t-s| > 1$.*

Remark 30. Direct computation yields

$$\frac{z_{s,t}''(x)}{z_{s,t}(x) + x} = \left[\frac{\psi(x+t) - \psi(x+s)}{t-s} \right]^2 + \frac{\psi'(x+t) - \psi'(x+s)}{t-s}. \quad (10.2)$$

To prove the positivity of the function (10.2), the following formula and inequality are used as basic tools in the proof of [9, Theorem 1]:

(1) For $x > -1$,

$$\psi(x+1) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x+k} \right). \quad (10.3)$$

(2) If $a \leq b < c \leq d$, then

$$\frac{1}{ab} + \frac{1}{cd} > \frac{1}{ac} + \frac{1}{bd}. \quad (10.4)$$

Remark 31. As consequences of Theorem 2, the following useful conclusions are derived:

(1) The function

$$e^{\psi(x+t)} - x \quad (10.5)$$

for all $t > 0$ is decreasing and convex from $(0, \infty)$ onto $(e^{\psi(t)}, t - \frac{1}{2})$.

(2) For all $x > 0$,

$$\psi'(x)e^{\psi(x)} < 1. \quad (10.6)$$

(3) For all $x > 0$ and $t > 0$,

$$\ln\left(x + t - \frac{1}{2}\right) < \psi(x+t) < \ln\left(x + e^{\psi(t)}\right). \quad (10.7)$$

(4) For $x > -\alpha$, the inequality

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} < \frac{t-s}{\psi(x+t) - \psi(x+s)} \quad (10.8)$$

holds if $|t-s| < 1$ and reverses if $|t-s| > 1$.

Remark 32. In fact, the function (10.5) is decreasing and convex on $(-t, \infty)$ for all $t \in \mathbb{R}$. See [35, Theorem 2].

Remark 33. It is clear that the double inequality (10.7) can be deduced directly from the decreasingly monotonic property of (10.5). Furthermore, from the decreasingly monotonic and convex properties of (10.5) on $(-t, \infty)$, the inequality (10.6) and

$$\psi''(x) + [\psi'(x)]^2 > 0 \quad (10.9)$$

on $(0, \infty)$ can be derived straightforwardly.

11. RECENT ADVANCES

Finally, we would like to state some new results related to or originated from Elezović-Giordano-Pečarić's Theorem 2 above.

11.1. Alternative proofs of Elezović-Giordano-Pečarić's theorem. The key step of verifying Theorem 2 is to prove the positivity of the right-hand side in (10.2) in which involves divided differences of the digamma and trigamma functions. The biggest barrier or difficulty to prove the positivity of (10.2) is mainly how to deal with the squared term in (10.2).

11.1.1. Chen's proof. In [5], the barrier mentioned above was overcome by virtue of the well-known convolution theorem [43] for Laplace transforms and so Theorem 2 for the special case $s + 1 > t > s \geq 0$ was proved. Perhaps this is the first try to provide an alternative of Theorem 2, although it was partially successful formally.

11.1.2. Qi-Guo-Chen's proof. For real numbers α and β with $(\alpha, \beta) \notin \{(0, 1), (1, 0)\}$ and $\alpha \neq \beta$, let

$$q_{\alpha, \beta}(t) = \begin{cases} \frac{e^{-\alpha t} - e^{-\beta t}}{1 - e^{-t}}, & t \neq 0, \\ \beta - \alpha, & t = 0. \end{cases} \quad (11.1)$$

In [39, 40], by making use of the convolution theorem for Laplace transform and the logarithmically convex properties of the function $q_{\alpha, \beta}(x)$ on $(0, \infty)$, an alternative proof of Theorem 2 was supplied.

11.1.3. Qi-Guo's proof. In [32], by considering monotonic properties of the function

$$Q_{s, t; \lambda}(u) = q_{\alpha, \beta}(u)q_{\alpha, \beta}(\lambda - u), \quad \lambda \in \mathbb{R} \quad (11.2)$$

and still employing the convolution theorem for Laplace transform, Theorem 2 was completely verified again.

Remark 34. For more information on the function $q_{\alpha, \beta}(t)$ and its applications, please refer to [26, 27, 30, 32, 36, 37] and related references therein.

11.1.4. Qi's proof. In [25, 29], the complete monotonic properties of the function in the right-hand side of (10.2) were established as follows.

Theorem 3. *Let s and t be two real numbers and $\alpha = \min\{s, t\}$. Define*

$$\Delta_{s, t}(x) = \begin{cases} \left[\frac{\psi(x+t) - \psi(x+s)}{t-s} \right]^2 + \frac{\psi'(x+t) - \psi'(x+s)}{t-s}, & s \neq t \\ [\psi'(x+s)]^2 + \psi''(x+s), & s = t \end{cases} \quad (11.3)$$

on $x \in (-\alpha, \infty)$. Then the functions $\Delta_{s, t}(x)$ for $|t - s| < 1$ and $-\Delta_{s, t}(x)$ for $|t - s| > 1$ are completely monotonic on $x \in (-\alpha, \infty)$.

Since the complete monotonicity of the functions $\Delta_{s, t}(x)$ and $-\Delta_{s, t}(x)$ mean the positivity and negativity of the function $\Delta_{s, t}(x)$, an alternative proof of Theorem 2 was provided once again.

One of the key tools or ideas used in the proofs of Theorem 3 is the following simple but specially successful conclusion: If $f(x)$ is a function defined on an infinite interval $I \subseteq \mathbb{R}$ and satisfies $\lim_{x \rightarrow \infty} f(x) = \delta$ and $f(x) - f(x + \varepsilon) > 0$ for $x \in I$ and some fixed number $\varepsilon > 0$, then $f(x) > \delta$ on I .

It is clear that Theorem 3 is a generalization of the inequality (10.9).

11.2. Complete monotonicity of divided differences. In order to prove the above Theorem 3, the following complete monotonic properties of a function related to a divided difference of the psi function were discovered in [29].

Theorem 4. *Let s and t be two real numbers and $\alpha = \min\{s, t\}$. Define*

$$\delta_{s,t}(x) = \begin{cases} \frac{\psi(x+t) - \psi(x+s)}{t-s} - \frac{2x+s+t+1}{2(x+s)(x+t)}, & s \neq t \\ \psi'(x+s) - \frac{1}{x+s} - \frac{1}{2(x+s)^2}, & s = t \end{cases} \quad (11.4)$$

on $x \in (-\alpha, \infty)$. Then the functions $\delta_{s,t}(x)$ for $|t-s| < 1$ and $-\delta_{s,t}(x)$ for $|t-s| > 1$ are completely monotonic on $x \in (-\alpha, \infty)$.

To the best of our knowledge, the complete monotonicity of functions involving divided differences of the psi and polygamma functions were investigated first in [23, 24, 25, 29].

11.3. Inequalities for sums. As consequences of proving Theorem 4 along a different approach from [29], the following algebraic inequalities for sums were procured in [23, 24] accidentally.

Theorem 5. *Let k be a nonnegative integer and $\theta > 0$ a constant.*

If $a > 0$ and $b > 0$, then

$$\sum_{i=0}^k \frac{1}{(a+\theta)^{i+1}(b+\theta)^{k-i+1}} + \sum_{i=0}^k \frac{1}{a^{i+1}b^{k-i+1}} > 2 \sum_{i=0}^k \frac{1}{(a+\theta)^{i+1}b^{k-i+1}} \quad (11.5)$$

holds for $b-a > -\theta$ and reverses for $b-a < -\theta$.

If $a < -\theta$ and $b < -\theta$, then inequalities

$$\sum_{i=0}^{2k} \frac{1}{(a+\theta)^{i+1}(b+\theta)^{2k-i+1}} + \sum_{i=0}^{2k} \frac{1}{a^{i+1}b^{2k-i+1}} > 2 \sum_{i=0}^{2k} \frac{1}{(a+\theta)^{i+1}b^{2k-i+1}} \quad (11.6)$$

and

$$\sum_{i=0}^{2k+1} \frac{1}{(a+\theta)^{i+1}(b+\theta)^{2k-i+2}} + \sum_{i=0}^{2k+1} \frac{1}{a^{i+1}b^{2k-i+2}} < 2 \sum_{i=0}^{2k+1} \frac{1}{(a+\theta)^{i+1}b^{2k-i+2}} \quad (11.7)$$

hold for $b-a > -\theta$ and reverse for $b-a < -\theta$.

If $-\theta < a < 0$ and $-\theta < b < 0$, then inequality (11.6) holds and inequality (11.7) is valid for $a+b+\theta > 0$ and is reversed for $a+b+\theta < 0$.

If $a < -\theta$ and $b > 0$, then inequality (11.6) holds and inequality (11.7) is valid for $a+b+\theta > 0$ and is reversed for $a+b+\theta < 0$.

If $a > 0$ and $b < -\theta$, then inequality (11.6) is reversed and inequality (11.7) holds for $a+b+\theta < 0$ and reverses for $a+b+\theta > 0$.

If $b = a - \theta$, then inequalities (11.5), (11.6) and (11.7) become equalities.

Moreover, the following equivalent relation between the inequality (11.5) and Theorem 4 was found in [23, 24].

Theorem 6. *The inequality (11.5) for positive numbers a and b is equivalent to Theorem 4.*

11.4. Recent advances. Recently, some applications, extensions and generalizations of the above Theorem 3, Theorem 4, Theorem 5 and related conclusions have been investigated in several coming published manuscripts such as [31, 34]. For example, Theorem 1 stated in Remark 15 has been obtained in [28].

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